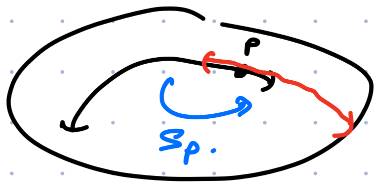
Lecture 29

Why symm spaces are automatically complete?

Hopf-Rinow: Complete $\Leftrightarrow$ every geodesic (locally len-min curve) can be extended to have domain $\mathbb{R}$.

If not: $\underset{\text{max domain}}{\longrightarrow}$ $\gamma^{-1}(p)$ $\xrightarrow{\gamma}$



Why transitive isom? $p \overset{m}{\bullet\!\!-\!\!\times\!\!-\!\!\bullet} q$. $s_m(p) = q$

Connection to decomp thm last time: $M \cong M_+ \times M_- \times M_0$ must assume M is simply connected.

Irreducible: Not a Riemannian product.

Irreducible simply conn sym spaces: $\mathbb{R}$ or G/K with G semisimple

either: G compact

or G not compact, K maximal cpt.

<u>Cartan decomp of a Lie alg.</u> $\mathfrak{g}$ $\mathbb{R}$ lie alg B Killing form (nondeg)

A Cartan invol is a map $\theta: \mathfrak{g} \to \mathfrak{g}$ that is a Lie alg aut. ($[\theta(x), \theta(y)] = [x,y]$) s.t. $\theta^2 = Id_{\mathfrak{g}}$ and $B^\theta(x,y) = -B(x, \theta(y))$ pos def.

e.g. G compact $\Rightarrow$ B is neg def $\Rightarrow$ $\theta = Id$ works.

<u>Thm</u>. Cartan involutions exist. For G s.c. they are in 1-1 corresp with maximal compact subgroups of G (or max subalg of G s.t. B neg def). The corresp is

$$\theta \rightsquigarrow \mathfrak{g} = \underset{+1}{\mathfrak{k}} \oplus \underset{-1}{\mathfrak{p}} \quad \theta\text{-eigenspace decomp} \longrightarrow \mathfrak{k} \rightsquigarrow K.$$

B-orthogonal.

<u>Ex</u>: On $\mathfrak{sl}_n\mathbb{R}$, $\theta(X) = -X^T$

RSP: (G, K, σ). In non compact type, $d\sigma_e = \theta$ the Cartan invol corresp to K.

Examples of riem. symm spaces:

Compact type.

1) $X = SO(n+1)/SO(n) = S^n$ w/ induced metric from $\mathbb{R}^{n+1}$ (invt $SO(n+1)$)

2) $X = SU(n+1)/SU(n) = \mathbb{C}P^n$ w/ Fubini-Study metric (the only metric on $\mathbb{C}P^n$ that is invt under $SU(n+1)$)

3) X = a compact Lie group L w/ a Riem met invt by left and right transl.

Then $G = \underbrace{L}_{L_g} \times \underbrace{L}_{R_{g^{-1}}} / Z(L)_\Delta \quad K = \Delta / Z(L)_\Delta \cong L/Z(L)$.

Exercise: Write $d\sigma_e : \mathfrak{g} \to \mathfrak{g}$ for these.

Noncompact Type

We can always take G/K and use $B|_\mathfrak{p}$ to get a metric.

1) Hyperbolic spaces. $\mathbb{H}^n = SO(n,1)/SO(n) \cong$ open ball in $\mathbb{R}^n$. ($\mathbb{H}^2$ = upper half plane)

2) Complex hyp spaces $\mathbb{C}\mathbb{H}^n = SU(n,1)/SU(n)$ ($\mathbb{C}\mathbb{H}^1$ = unit disk)
$\cong$ open ball in $\mathbb{C}^n$.

3) $\mathrm{Inner}_0(\mathbb{R}^n) = \{$positive definite inner prod s.t. $[-1,1]^n$ has volume $2^n\}$
Such an inner product always has the form $\langle x, y\rangle_A = (Ax)^T(Ay)$
$= x^T A^T A x$
for $A \in SL_n\mathbb{R}$. A is not unique because if $M^T M = Id$, then A and MA have the same assoc $\langle , \rangle$.

So $\langle , \rangle_A \longmapsto SO(n)\cdot A$ gives $\mathrm{Inner}_0(\mathbb{R}^n) \cong SO(n) \backslash SL_n\mathbb{R}$

The action: $(T \cdot \langle , \rangle_A)(x,y) = \langle Tx, Ty \rangle_A$ is a <u>right</u> action

$\left(S \cdot (T \cdot \langle , \rangle_A)\right)(x,y) = (T \cdot \langle , \rangle_A)(Sx, Sy) = \langle TSx, TSy \rangle_A$

$= ((TS) \cdot \langle , \rangle_A)(x,y)$.

<u>Fix</u>. Define $\langle , \rangle_A$ using A^{-1}. $\quad$ $\mathrm{Inner}_0 \cong SL_n\mathbb{R}/SO(n)$

$T \cdot \langle , \rangle_A = \langle T^{-1}x, T^{-1}y \rangle$ $\quad$ w/ left mul by $SL_n\mathbb{R}$

Cartan: $SL_n\mathbb{R} = \mathfrak{k} \oplus \mathfrak{p}$ $\quad$ $\mathfrak{k} = \{X + X^T = 0\}$ $\quad$ $\mathfrak{p} = \{X - X^T = 0\}$
$X = X^T$.

$B(X,Y) = \mathrm{tr}(XY)$ up to scale. Check: neg def on $\mathfrak{k}$ as

$\mathrm{tr}(XY) =$ sum of (row i x)$\cdot$(col i of y) $\quad$ if $X = -X^T$

$\mathrm{tr}(XX) =$ sum ———————— $= -$(row i)$\cdot$(row i) ≤ 0.

Metric interp: Given an inner product $\langle , \rangle = p$.

$v =$ tgl vec of $\langle (I - \varepsilon X)\cdot, (I - \varepsilon X)\cdot \rangle$ where $X = X^T$

$w =$ ———————— $\langle (I - \varepsilon Y)\cdot, (I - \varepsilon Y)\cdot \rangle$ $\quad$ $Y = Y^T$.

Then $g_p(v,w) = \mathrm{tr}(XY)$.

<u>Exercise</u>. $\gamma(t) =$ inner product where e_1, e_2, e_3 orthog.

$\|e_1\| = a(t)$ $\quad$ $\|e_2\| = b(t)$ $\quad$ $\|e_3\| = \frac{1}{a(t)\,b(t)}$.

Find length $\gamma(t)$ $t \in [0,1]$.

4) $\mathrm{Herm}(\mathbb{C}^n) = \{$hermitian, pos def inner on $\mathbb{C}^n$ s.t. polydisk Δ^n has volume $\pi^n\}$

$\cong SL_n\mathbb{C}/SU(n)$ $\quad$ by $\langle z, w \rangle = (\overline{A^{-1}z})^t (A^{-1}w) \longleftrightarrow A\, SU(n)$

$\mathfrak{sl}_n\mathbb{C} = \mathfrak{k} \oplus \mathfrak{p} \quad \mathfrak{p} = i\mathfrak{k}$ real part of the complex killing form.

Low dim accident: $PSL_2\mathbb{C} \cong SO_0(3,1) \rightsquigarrow \underbrace{\mathrm{Herm}(\mathbb{C}^2)}_{\text{Hermitian matrix model.}}$ iso to $\underbrace{\mathbb{H}^3}_{\text{Minkowski model}}$.

Curvature

$R_p(v,w)$ is a linear operator on T_pM (Riem curv tensor)

On a symm space, if we identify tangent spaces with $\mathfrak{p}$ up to $\mathfrak{k}$,

$$R(x,y) = \mathrm{ad}_{\underbrace{[x,y]}_{\mathfrak{k}}} : \mathfrak{p} \to \mathfrak{p}.$$

i.e. $g(R(x,y)z, w) = B([[x,y],z], w)$

Has positive curvature: $g(R(v,w)(v),w) > 0 \quad \forall v, w$

etc.

For G/K noncpt type: $B([[x,y],x], y)$

$$[[x,y],x] + \cancel{[[x,x],y]} + [[y,x],x] = 0$$

$$[[x,y],x] = -[[y,x],x] = [x,[y,x]] = -[x,[x,y]]$$

$$B([[x,y],x],y) = -B([x,[x,y]],y) = -B(\mathrm{ad}_x[x,y], y)$$

$$= B([x,y], \mathrm{ad}_x(y)) = B([x,y],[x,y])$$

Now $[x,y] \in \mathfrak{k}$ so this is ≤ 0.

Cor Symm spaces of noncpt type have nonpos curvature.

Cor. Symm sp of noncpt type diffeo to $\mathbb{R}^n$.